GFD I, Solutions for Problem Set #4, 2/29/2012

[30 points total]

For reference, the governing equations for this problem are:

$$\text{X-MOM}_1 \qquad -fv_1 = -g\eta_x$$

$$\text{X-MOM}_2 \qquad 0 = -g\eta_x - g'E_x$$

(a) [2 pts.] Since the flow is geostrophic, the surface height is related to the upper layer velocity by: $\eta = \int \left(f/g\right) v_1\, dx$, and so

$$\eta = \frac{f}{kg} V \sin\left(kx\right).$$

(b) [3 pts.] The interface shape can be seen from X-MOM$_2$ to be $E = -\left(g/g'\right)\eta$, so

$$E = -\frac{f}{kg'} V \sin\left(kx\right).$$

The maximum value of this, for the parameter values given, is:

$$E_{\max} = \frac{fV}{g'k} = \frac{\left(10^{-4}\ \text{s}^{-1}\right)\left(0.5\ \text{m s}^{-1}\right)}{\left(10^{-2}\ \text{m s}^{-2}\right)\left(\pi \times 10^{-4}\ \text{m}^{-1}\right)} = 16\ \text{m}.$$

(c) [10 pts.] The expression for the kinetic energy per unit horizontal area at this level of approximation is given by

$$KE_A = \frac{1}{2}\rho_1 H_1\left(u_1^2 + v_1^2\right) + \frac{1}{2}\rho_1 H_2\left(u_2^2 + v_2^2\right) = \frac{1}{2}\rho_1 H_1 v_1^2.$$

Note that within the Boussinesq approximation you can use some representative value for the density in this expression – I have called it ρ_1, but it could be ρ_2, or their average – they only differ by 0.1%. For the potential energy we have to be more careful, because even within the Boussinesq approximation the density perturbations play a role in the vertical momentum equation where they are multiplied by gravity. Thus the potential energy per unit horizontal area is calculated as:

$$
\begin{aligned}
PE_A &= \int_{-H}^{\eta} \rho g z\, dz = \int_{-H}^{-H_1+E} \rho_2 g z\, dz + \int_{-H_1+E}^{\eta} \rho_1 g z\, dz \\
&= \frac{1}{2} g\left(\rho_1 + \Delta\rho\right)\left[\left(-H_1 + E\right)^2 - H^2\right] + \frac{1}{2} g\rho_1\left[\eta^2 - \left(-H_1 + E\right)^2\right] \\
&= \frac{1}{2} g\Delta\rho\left(-H_1 + E\right)^2 - \frac{1}{2} g\rho_2 H^2 + \frac{1}{2} g\rho_1 \eta^2
\end{aligned}
$$

Then the rest-state potential energy will be given by taking E and η to be zero:

$$PE_{A0} = \frac{1}{2} g\Delta\rho\left(-H_1\right)^2 - \frac{1}{2} g\rho_2 H^2 .$$

Subtracting this from the full expression gives the desired expression:

$$APE_A = PE_A - PE_{A0} = \frac{1}{2} g\Delta\rho\left[E^2 - 2H_1 E\right] + \frac{1}{2} g\rho_1 \eta^2 .$$

(d) [5 pts.] Taking the *x*-averages of our energy expressions, and forming their ratio, we find:

$$
\begin{aligned}
\overline{KE_A}^x &= \frac{1}{2}\rho_1 H_1 \overline{v_1^2}^x = \frac{1}{4}\rho_1 H_1 V^2 \\
\overline{APE_A}^x &= \frac{1}{2} g\Delta\rho \overline{E^2}^x - g\Delta\rho H_1 \cancel{\overline{E}^x} + \frac{1}{2} g\rho_1 \overline{\eta^2}^x \\
&= \frac{1}{4}\left(\frac{g\Delta\rho}{\rho_1}\right)\rho_1 \frac{f^2 V^2}{g'^2 k^2} + \frac{1}{4} g\rho_1 \frac{f^2 V^2}{g^2 k^2} \\
&= \frac{1}{4}\rho_1 \frac{f^2 V^2}{g' k^2} + \frac{1}{4}\rho_1 \frac{f^2 V^2}{g k^2} \\
\frac{\overline{APE_A}^x}{\overline{KE_A}^x} &= \frac{f^2}{H_1 g' k^2} + \frac{f^2}{H_1 g k^2}
\end{aligned}
$$

(e) [5 pts.] Simplifying the final ratio for $g >> g'$ gives

$$\frac{\overline{APE_A}^x}{\overline{KE_A}^x} \cong \frac{f^2}{H_1 g' k^2} = \frac{1}{\left(ka'\right)^2}$$

where we have used the "internal" Rossby radius of deformation in the limit of a very thick lower layer:

$$a' = \frac{\sqrt{g' \dfrac{H_1 H_2}{H_1 + H_2}}}{f} \underset{\lim H_2 \to \infty}{=} \frac{\sqrt{g' H_1}}{f} .$$

The value of a' in this case is:

$$a' = \frac{\sqrt{\left(10^{-2}\ \text{m s}^{-2}\right)\left(10^{2}\ \text{m}\right)}}{10^{-4}\ \text{s}^{-1}} = \frac{1\ \text{m s}^{-1}}{10^{-4}\ \text{s}^{-1}} = 10\ \text{km}\,.$$

Note that this is much smaller than the "external" Rossby radius, which we defined when looking at the Shallow Water Equations. The internal Rossby radius is much more relevant to atmospheric and oceanic motions (except tides) because much of their energy is in baroclinic modes. Thus, for motion with length scale greater than a', we expect rotation effects to be important, and the flow to be dominated by winds or currents in geostrophic balance.

(f) [5 pts.] From the results of (e) we may state that

$$\frac{\overline{APE_A}^{x}}{\overline{KE_A}^{x}} \cong \frac{1}{(ka')^2} = \left(\frac{L}{a'}\right)^2$$

which will clearly be dominated by APE when the length scale becomes large relative to the internal Rossby radius. The potential energy will be almost entirely associated with the interface displacements, which govern the first term on the RHS of:

$$\overline{APE_A}^{x} = \frac{1}{2} g \Delta\rho \overline{E^2}^{x} + \frac{1}{2} g \rho_1 \overline{\eta^2}^{x} = \underbrace{\frac{1}{4}\rho_1 \frac{f^2V^2}{g'k^2}}_{\text{big}} + \underbrace{\frac{1}{4}\rho_1 \frac{f^2V^2}{gk^2}}_{\text{negligible}}\,.$$